# A MODEL FOR CHARGES OF ELECTROMAGNETIC TYPE

Detlev Buchholz, Sergio Doplicher, Gianni Morchio,
John E. Roberts, Franco Strocchi

Institut für Theoretische Physik der Universität Göttingen,
D-37073 Göttingen, Germany
Dipartimento di Matematica, Università di Roma "La Sapienza",
I-00185 Roma, Italy
Dipartimento di Fisica dell'Università, I-56126 Pisa, Italy,
Dipartimento di Matematica, Università di Roma "Tor Vergata",
I-00133 Roma, Italy
Scuola Normale Superiore, I-56126 Pisa, Italy

## 1. Introduction

Despite its successes, the theory of superselection sectors still needs extending if it is to cover all cases of physical interest. Even if we restrict our attention to four spacetime dimensions, the two cases that have been treated successfully, localized charges [1] and cone–localized charges [2], do not include the physically relevant class of theories with a local Abelian gauge group, notably quantum electrodynamics.

The insights into the superselection structure of the latter theories gained so far, basically have the character of no–go–theorems. We recall in this context that, as a consequence of Gauss' law, electrically (or magnetically) charged superselection sectors

(a) are not invariant under Lorentz transformations [3], [4],
(b) cannot be generated from the vacuum by field operators localized in bounded spacetime regions [1], [5] or arbitrary spacelike cones [6], and
(c) do not admit a conventional particle interpretation (infraparticle problem) [3], [4].

Consequently, it does not seem possible to characterize these superselection sectors by the spectrum (dual) of some group of internal symmetries nor to assign to each sector some definite statistics. Whereas in massive theories, this is always possible [7].

The aim of the present article is to discuss a simple but instructive model showing that the situation is actually not as hopeless as it may seem. Even though Gauss' law holds for the charged states in this model with all its inevitable consequences, the corresponding sectors can be labelled by the spectrum of some internal symmetry group and have well defined statistics. More

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

interestingly, the properties of these charged states seem to point to a general argument allowing one to establish these features for all charges of electric and magnetic type. Our present findings thus set the stage for a more general analysis of this issue to be carried out elsewhere.

Prior to discussing our model, let us explain in heuristic terms why these rather unexpected results are in harmony with the above no–go–theorems. First, the Coleman–Mandula Theorem apparently says that the charged superselection sectors cannot be characterized by an internal symmetry group for it states [8] that (bosonic) internal symmetries and geometric symmetries decouple, in obvious conflict with (a). However, as there is no proper scattering matrix for the charged states, cf. (c), the Coleman–Mandula Theorem does not apply here. Hence there is room for Lorentz transformations to act non–trivially on internal symmetries.

Secondly, as the charged states cannot be localized adequately, cf. (b), a proper definition of statistics of the corresponding sectors seems impossible. The loopholes are not so obvious here. The essential point is that, because of (a), any charged state singles out a Lorentz frame. Hence one must expect observations in that frame to play a special role. Phrased differently, some distinguished set of observables should allow one to identify that particular frame. Obviously, this set of observables cannot be stable under Lorentz transformations.

The following hypothesis seems attractive: a charged state singles out the observables on which it is well localized. These distinguished observables include (tensorial) charge densities in some specific Lorentz frame. Indeed, in the Coulomb gauge of quantum electrodynamics the charged Fermi field commutes at equal times and finite distances with the charge density and the magnetic field, but not with the spatial components of the current and the electric field [9]. So there is some evidence that the electric charge is, in this restricted sense, localizable; further support for this idea will be provided by our model. The effects of exchanging charges localized in disjoint regions can still be analyzed with this restricted notion, thereby providing a basis for discussing statistics.

The idea that electric charges can be localized in the restricted sense indicated above was first put forward by Fröhlich [10], who based a general analysis of superselection sectors in quantum electrodynamics on this assumption. Yet the problem of statistics was not discussed in that article. Moreover, some of the technical assumptions made in the analysis seem physically unreasonable, cf. the remarks below. To ensure a consistent setting, the present article focuses on a concrete model illustrating the expected subtle features of the theories of interest here.

For the convenience of the reader unfamiliar with the algebraic setting of quantum field theory [11] and the more recent developments in the theory of superselection sectors [7] we use the remainder of this introduction to describe our model in basic field–theoretic terms commenting on the significance of various steps in our analysis. The expert reader might wish to proceed directly to the subsequent section.

For our model we draw on the theory of a free, massless, scalar field $\phi$ in $s{+}1$ spacetime dimensions in the vacuum Hilbert (Fock) space $\mathcal{H}_0$. This requires some comment: we are interested in long range effects mediated by low energy

excitations but believe that interactions can be neglected without distorting the general qualitative picture since, according to well–known low energy theorems, low energy excitations do not interact with each other [12]. However, we use a scalar field $\phi$ merely as a matter of notational convenience. Similar models can be based on the free electromagnetic field.

The free, massless, scalar field $\phi$ is known to have many superselection sectors with different infrared properties. We are interested here in sectors distinguished by a tensorial charge, cf. (a), where Gauss' law holds. The simplest tensor field constructed from $\phi$ and yielding such a charge is

$$F_{\mu\nu\,(\lambda)}(x) = (\partial_\mu g_{\nu\lambda} - \partial_\nu g_{\mu\lambda})\phi(x), \tag{1.1}$$

where $\partial_\mu$ are the spacetime derivatives and $g_{\nu\lambda}$ is the metric. Since $\Box\phi = 0$, the corresponding identically conserved current is given by

$$j_{\mu\,(\lambda)}(x) = \partial^\nu F_{\mu\nu\,(\lambda)}(x) = \partial_\mu\partial_\lambda\phi(x). \tag{1.2}$$

Because of Gauss' law and locality of the field $\phi$, the resulting charge operators $Q_{(\lambda)}$ are clearly 0 on the vacuum Hilbert space $\mathcal{H}_0$.

To describe charged states, we have to change the representation of the algebra generated by the field $\phi$. This can most easily be done with the help of automorphisms $\gamma$ acting on polynomials in the field $\phi$. We put

$$\gamma(\phi(x)) := \phi(x) + \varphi(x)\,1, \tag{1.3}$$

where $\varphi$ is any real distributional solution of the wave equation. The action of $\gamma$ on arbitrary polynomials in $\phi$ is obtained from (1.3) by linearity and multiplicativity. In particular, $\gamma$ acts as follows on charge densities:

$$\gamma(j_{0\,(\lambda)}(x)) = j_{0\,(\lambda)}(x) + \partial_0\partial_\lambda\varphi(x)\,1. \tag{1.4}$$

Let us now pursue the idea that the charge density of the states described by $\gamma$ is well localized. This leads us to take the Cauchy data of $\varphi$ to satisfy the following condition:

$$\triangle\varphi(0,\vec{x}) = \rho(\vec{x}), \quad (\partial_0\varphi)(0,\vec{x}) = \sigma(\vec{x}), \tag{1.5}$$

where $\rho, \sigma$ have compact support. Then, as $\varphi$ propagates causally, $\gamma$ will act like the identity on $j_{0\,(\lambda)}(x)$ for $x$ in the causal complement of a bounded spacetime region. To simplify the subsequent discussion, we assume that $\rho, \sigma$ are smooth and compute

$$\int d^s\vec{x}\,\partial_0\partial_\lambda\varphi(x) = \begin{cases} \int d^s\vec{x}\,\rho(\vec{x}) & \lambda = 0 \\ 0 & \lambda \neq 0. \end{cases} \tag{1.6}$$

It would be premature to infer that $\gamma$ describes charged states if $\int d^s\vec{x}\,\rho(\vec{x}) \neq 0$, since the operations of integrating over an infinite volume and acting with $\gamma$ cannot simply be interchanged. Indeed, a more careful analysis shows that $\gamma$ does not lead to a new superselection sector if $s > 3$, i.e., the symmetry associated with the current is spontaneously broken in these cases. Whilst there is an associated Gauss law, the spontaneous breakdown is not accompanied by

a mass gap in the energy–momentum spectrum, despite some of the folklore on the Higgs mechanism. Yet if $s \leq 3$, $\gamma$ describes charged states of the charges $Q_{(\lambda)}$ given in (1.6).

If $s \geq 2$, the charged sectors turn out to be invariant under spacetime translations but of course not under Lorentz transformations. If $s \geq 3$, translations satisfy the relativistic spectrum condition. Hence if $s = 3$, the model exhibits all desirable features and we restrict our attention to this case in the following.

To discuss the statistics of charged sectors, we have to consider states whose charge densities are localized in different regions. They are obtained by translating the automorphisms $\gamma$, setting

$$\gamma_a(\phi(x)) := \phi(x) + \varphi(x-a)\,1 \tag{1.7}$$

for arbitrary spacetime translations $a$. For the moment let us assume for pedagogic reasons that there are unitary "charged field operators" $W_a$, acting on a suitable extension of $\mathcal{H}_0$ to a Hilbert space of charged states and implementing the automorphisms $\gamma_a$,

$$\gamma_a(\phi(x)) = W_a^{-1}\phi(x)W_a. \tag{1.8}$$

(It is not difficult to see that such operators exist in the present model in certain non–regular representations of some Weyl–algebra, cf. [13]. We will elaborate on this in the main text.) The statistics of the charged sectors can then be read off from the behaviour of their commutators, or better from their group–theoretic commutator $\varepsilon_{ab} = W_a^{-1}W_b^{-1}W_aW_b$, in the limit of large spacelike separation of $a$ and $b$.

In fact, the statistics operator $\varepsilon_{ab}$ can be computed without first having to solve the more difficult problem of constructing charged field operators and this is important for a general structural analysis, cf. the case of strictly localized charges [1]. To indicate how this can be done and the type of problems that arise, let us rewrite $\varepsilon_{ab}$ in the form

$$\varepsilon_{ab} = (W_a^{-1}W_b)\,W_b^{-1}(W_b^{-1}W_a)W_b. \tag{1.9}$$

The operator $U_{ab} = W_a^{-1}W_b$ appearing in this expression intertwines the automorphisms $\gamma_b$ and $\gamma_a$, i.e.,

$$U_{ab}\,\gamma_b(\phi(x))\,U_{ab}^{-1} = \gamma_a(\phi(x)), \tag{1.10}$$

and can obviously be interpreted as transporting charge. In the present model these charge transporters are defined on the vacuum Hilbert space $\mathcal{H}_0$, but, in contrast to the case of strictly localizable charges, they cannot be approximated by local operators in the norm topology. Such an assumption was made in [10], but does not seem consistent with gauge theories.

Returning to (1.9), we see that the expression following $U_{ab}$ involves the adjoint action of $W_b^{-1}$, an action, coinciding with that of the automorphism $\gamma_b$ on local operators on $\mathcal{H}_0$, according to (1.8). Yet here it acts on $U_{ba} = U_{ab}^{-1}$, so the question arises of whether the charge–carrying automorphisms can be extended to the charge transporters. If so, one could represent $\varepsilon_{ab}$ in the form

$\varepsilon_{ab} = U_{ab}\,\gamma_b(U_{ab}^{-1})$ and the charged fields would have been completely replaced by quantities intrinsically defined on the vacuum sector $\mathcal{H}_0$.

The physical idea that $\gamma_b$ arises by shifting a charge from infinity (where it has no effect) to $b$ suggests extending the automorphism $\gamma_b$ by setting:

$$\gamma_b(U) := \lim_c U_{bc} U U_{bc}^{-1}, \tag{1.11}$$

as $c$ tends to spacelike infinity. So the question arises of whether this limit exists (in norm) for all charge transporters $U$, independent of the choice of the sequence $c$. As the charge transporters factorize: $U_{bc}\,U_{cd} = U_{bd}$, the answer to this question is encoded in their asymptotic commutation properties. In the model at hand, the relevant conditions are satisfied, enabling one to define $\varepsilon_{ab}$, the first important step in discussing statistics.

The next problem is whether $\varepsilon_{ab}$ converges to a limit independent of $a$ as $b$ tends spacelike to infinity, as expected if there are charged fields with definite asymptotic commutation relations. What matters here is how the charged automorphisms act on intertwiners transporting charges between distant regions. If this action becomes trivial,

$$\lim_{c,d} \big(\gamma_b(U_{cd}) - U_{cd}\big) = 0 \tag{1.12}$$

as $c, d$ tend spacelike to infinity, then the above limit of $\varepsilon_{ab}$ exists and has the desired properties. This condition turns out to be satisfied in the present model.

To determine the nature of the statistics of the charged sectors, one further step is needed. Instead of sending $b$ in $\varepsilon_{ab}$ spacelike to infinity, keeping $a$ fixed, one can interchange the role of $a$ and $b$. If the limit is the same, the charged sectors have permutation group statistics (and not just braid group statistics). With simple sectors, generated by automorphisms, as here, the only remaining possibilities are Bose and Fermi statistics. An explicit computation in our model shows that $\lim_a \varepsilon_{ab} = \lim_b \varepsilon_{ab} = 1$, implying Bose statistics.

This outline of our results makes it clear that the relevant objects when discussing statistics are the charged automorphisms and their intertwiners (charge transporters), just as in the case of localized charges. The crucial additional information is the asymptotic commutation properties of the intertwiners, cf. relations (1.11) and (1.12). As a matter of fact these properties even suffice to go further and to establish systematically that charged field operators and a global gauge group exist in our model. The natural mathematical setting for discussing these issues is the theory of tensor categories, developed in [14]. We will therefore use that setting in the main text.

## 2 The Model

The aim of this section is to give a precise definition of our model. It will then be analyzed in subsequent sections and finally we will comment on those aspects which transcend the particular features of our model.

Our model is defined in terms of Weyl operators and the sectors will be constructed using automorphisms generated by Weyl operators.

We use standard notation for the Weyl commutation relations:

$$W(f)W(f') = e^{\frac{i}{2}\sigma(f,f')}W(f+f'), \tag{2.1}$$

where $\sigma$ is a symplectic form. Our model is just the free massless scalar field in $s$ space dimensions, $s \geq 2$. But we will now confine our attention to the case $s = 3$ as we have already commented in the introduction on how the relevant features of the model depend on $s$. Expressing the field in terms of the Cauchy data at time $t = 0$, we may take the underlying space to be

$$\mathcal{L} := \omega^{-\frac{1}{2}}\mathcal{D}(\mathbb{R}^s) + i\omega^{\frac{1}{2}}\mathcal{D}(\mathbb{R}^s). \tag{2.2}$$

Here $\mathcal{D}(\mathbb{R}^s)$ denotes the space of smooth, real–valued functions with compact support and $\omega$ the energy operator. The symplectic form is given by

$$\sigma(f,f') = -\mathrm{Im}\langle f,f'\rangle, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the scalar product on $\mathcal{L}$ determining the usual vacuum state, i.e. the one with vanishing one–point functional:

$$\langle f,f'\rangle := \int d^s\vec{x}\,\overline{f(\vec{x})}f'(\vec{x}). \tag{2.4}$$

The resulting net of von Neumann algebras in the vacuum representation will be denoted by $\mathfrak{A}$.

Our charges will turn out to be localized on the subnet $\mathfrak{A}_0$ which we specify by giving the appropriate subspace $\mathcal{L}_0$ of $\mathcal{L}$:

$$\mathcal{L}_0 := \omega^{\frac{3}{2}}\mathcal{D}(\mathbb{R}^s) + i\omega^{\frac{1}{2}}\mathcal{D}(\mathbb{R}^s). \tag{2.5}$$

Notice that these functions are less singular at the origin in momentum space. By contrast, to define the automorphisms giving rise to the sectors in the model, we choose a space $\mathcal{L}_\Gamma$ of functions more singular at the origin:

$$\mathcal{L}_\Gamma := \omega^{-\frac{1}{2}}\mathcal{D}(\mathbb{R}^s) + i\omega^{-\frac{3}{2}}\mathcal{D}(\mathbb{R}^s). \tag{2.6}$$

Since $\mathcal{L}$ is invariant under spacetime translations and $\mathcal{L}_0 = i\omega\mathcal{L}$, $\mathcal{L}_0$ is also invariant under spacetime translations. It is not, however, invariant under Lorentz boosts. In fact, multiplying by $i\omega$ is an isomorphism from $\mathcal{L}$ to $\mathcal{L}_0$ considered in the obvious way as nets at time $t = 0$. This isomorphism induces an isomorphism of nets of von Neumann algebras. Similarly, since $\mathcal{L}_\Gamma = i\omega^{-1}\mathcal{L}$, $\mathcal{L}_\Gamma$ is invariant under spacetime translations.

Of course, $\mathcal{L}_\Gamma$ only specifies automorphisms of $\mathfrak{A}$ if our symplectic form has been extended in the first variable to $\mathcal{L}_\Gamma$. In fact, our symplectic form $\sigma$ extends in a natural way to $\mathcal{L}_\Gamma$. In terms of the smooth functions $g$, $h$ parameterizing $\gamma$,

$$\gamma = i\omega^{-\frac{3}{2}}g + \omega^{-\frac{1}{2}}h, \quad g,h \in \mathcal{D}(\mathbb{R}^s), \tag{2.7}$$

we have

$$\sigma(\gamma,\gamma') = \int d^s\vec{p}\,\omega^{-2}(\tilde{g}(-\vec{p})\tilde{h}'(\vec{p}) - \tilde{g}'(-\vec{p})\tilde{h}(\vec{p})). \tag{2.8}$$

Seen from the point of view of a Weyl algebra, extending to $\mathcal{L}_\Gamma$ in this way would give a Weyl algebra with a non–regular vacuum state, cf. [13]. It proves convenient to use the symbol $\gamma$ to denote the automorphism generated by $\gamma \in \mathcal{L}_\Gamma$ so that

$$\gamma(W(f)) = e^{-i\mathrm{Im}\langle \omega^{\frac{1}{2}}\gamma,\omega^{-\frac{1}{2}}f\rangle}\, W(f). \tag{2.9}$$

$\Gamma$ will denote the corresponding group of automorphisms. Notice that if $\gamma = \omega^{-\frac{1}{2}}h$, $h \in \mathcal{D}(\mathbb{R}^s)$ then $\gamma \in L^2(\mathbb{R}^s)$ so that such $\gamma$ will not lead to a new sector. The automorphisms in question extend to the local von Neumann algebras. As $s$ is odd, this follows from Huygens principle, for $\sigma(e^{it\omega}i\omega\gamma, f) = 0$ for $t > t_0$, where $t_0$ depends only on supp$f$. Thus we have

$$\sigma(\gamma, f) = -\int_0^{t_0} dt\,\sigma(e^{it\omega}i\omega\gamma, f) \tag{2.10}$$

and $\int_0^{t_0} dt\, e^{it\omega}i\omega\gamma \in L^2(\mathbb{R}^s)$.

We regard the automorphisms in $\Gamma$ as defining representations of the observable net $\mathfrak{A}$. These representations are covariant and satisfy the spectrum condition. Their unitary equivalence classes will be our sectors. For this reason, we define two automorphisms $\gamma$, $\gamma' \in \Gamma$ to be equivalent if $\gamma'\gamma^{-1}$ is induced by a unitary on the vacuum Hilbert space. This is equivalent to saying that $\gamma' - \gamma \in \mathcal{L}_\Gamma$ is in the 1–particle space $L^2(\mathbb{R}^s)$. In terms of the smooth functions $g, h$ and $g', h'$ parametrizing $\gamma$ and $\gamma'$ we have equivalence if and only if $g - g'$ has a zero at zero in momentum space, i.e. if and only if

$$\int g\, d^s\vec{x} = \int g'\, d^s\vec{x}. \tag{2.11}$$

For us the important property of these automorphisms is that they are localized on $\mathfrak{A}_0$ considered as a net over double cones with axis in the time–direction. It suffices to verify that they are localized in double cones at time $t = 0$, and, because of the causal propagation properties, it suffices to show that given $\gamma \in \Gamma$, there is a ball $B$ at time $t = 0$ such that

$$\gamma(W(f)) = W(f), \tag{2.12}$$

whenever the smooth functions determining $f \in \mathcal{L}_0$ have support in the complement of $B$. But this follows from

$$\sigma(\gamma, f) = \sigma(i\omega\gamma, i\omega^{-1}f). \tag{2.13}$$

Another property of interest is *translatability*. An automorphism $\gamma$ of $\mathfrak{A}$ is said to be translatable if it is equivalent to its translates, i.e. if $\gamma_a := \alpha_a\gamma\alpha_{-a}$ and $\gamma$ are equivalent for each spacetime translation $a$. Translating $\gamma$ corresponds to translating $g$ and $h$ so translatability follows from the criterion for

equivalence derived above. As we can dilate the Cauchy data without changing the equivalence class, our charged automorphisms are *transportable* in the sense that there are equivalent automorphisms localized over $\mathfrak{A}_0$ in any double cone with axis in the time–direction. In fact, as $s = 3$, this is even achieved by the canonical automorphic action of the dilation group on $\mathfrak{A}$ since the induced action on $\Gamma$ is given by:

$$g_\lambda(\vec{x}) := \lambda^{-\frac{s+3}{2}}\, g(\lambda^{-1}\vec{x}), \quad h_\lambda(\vec{x}) := \lambda^{-\frac{s+1}{2}} h(\lambda^{-1}\vec{x}). \tag{2.14}$$

Our sectors are thus dilation and translation invariant.

Finally, by virtue of Gauss' law applied to the current (1.2), we expect to be able to determine the charge carried by the automorphisms $\gamma$ by testing them with suitable observables, eventually localized in the spacelike complement of any given double cone in Minkowski space $\mathbb{R}^{s+1}$. In fact, pick some function $k \in \mathcal{D}(\mathbb{R}^s)$ equal to $\kappa$ on the unit ball centred at the origin and put $f_\lambda := \omega^{3/2}\, k_\lambda$, $\lambda > 0$, where $k_\lambda(\vec{x}) := k(\lambda^{-1}\vec{x})$. $\omega^2\, k_\lambda$ has support in the complement of a ball of radius $\lambda$ centred at the origin so the Weyl operator $W(f_\lambda)$, which results from $W(f_1)$ by the action of dilations, is localized in the spacelike complement of that ball. Hence $W(f_\lambda)$ is a "central sequence" commuting with all operators in $\mathfrak{A}$ in the limit $\lambda \to \infty$ and the (non–vanishing) vacuum expectation values $\omega_0(W(f_\lambda))$ do not depend on $\lambda$. Consequently

$$W(f_\lambda) \to \omega_0(W(f_1))\, 1 \tag{2.15}$$

in the weak operator topology in this limit. On the other hand, taking say $\gamma = i\omega^{-\frac{3}{2}} g$, with $g \in \mathcal{D}(\mathbb{R}^s)$, we obtain

$$\sigma(\gamma, f_\lambda) = \int d^s\vec{x}\, g(\vec{x}) k(\lambda^{-1}\vec{x}) \tag{2.16}$$

implying

$$\gamma(W(f_\lambda)) = e^{i\sigma(\gamma,f_\lambda)}\, W(f_\lambda) \to e^{i\kappa q}\, \omega_0(W(f_1))\, 1 \tag{2.17}$$

in the weak operator topology, where, in virtue of (1.2), $q := \int d^s\vec{x}\, g$ may be identified as the charge carried by $\gamma$. Thus inequivalent automorphisms remain inequivalent on restriction to the observables localized in the spacelike complement of any double cone. So, as expected, our sectors do not satisfy the selection criterion of [1].

We sum up the results of this section as follows.

**Theorem 1.** *The automorphisms in $\Gamma$ are localized over $\mathfrak{A}_0$ and transportable forming a group stable under translations and dilations. The corresponding sectors are invariant under translations and dilations and do not satisfy the selection criterion of [1].*

In the next section we consider intertwiners between these automorphisms.

## 3 Intertwiners

Given $\gamma$, $\delta \in \Gamma$, we write $T \in (\gamma, \delta)$ to denote a bounded linear operator on the vacuum Hilbert space such that

$$T\gamma(A) = \delta(A)T, \quad A \in \mathfrak{A}. \tag{3.1}$$

$T$ is said to intertwine $\gamma$ and $\delta$. As we are dealing with automorphisms and $\mathfrak{A}$ is irreducible, $(\gamma, \delta)$ is at most one dimensional and, if non–zero, consists of multiples of a unitary operator. In this case, $\delta\gamma^{-1}$ is unitarily implementable in the vacuum Hilbert space and is hence implemented by a unique Weyl operator (see e.g. Lemma A.1 of [15]), namely $W(\delta - \gamma)$.

We regard the set of intertwiners as the arrows in a category, the objects then being the automorphisms. Composing arrows means multiplying the underlying operators and we use the symbol $\circ$ to denote this composition whenever emphasis is necessary.

We are interested in the asymptotic commutation properties of intertwiners and note that if $U \in (\gamma, \delta)$ and $U' \in (\gamma', \delta')$ are non-zero intertwiners then their group–theoretical commutator $UU'U^{-1}U'^{-1}$ is a phase, $\eta(\gamma, \delta; \gamma', \delta')$, independent of the choice of $U$ and $U'$. Taking $U$ and $U'$ to be Weyl operators, we see that

$$\eta(\gamma, \delta; \gamma', \delta') = e^{i\sigma(\delta-\gamma, \delta'-\gamma')}. \tag{3.2}$$

As our symplectic form $\sigma$ has been extended to $\mathcal{L}_\Gamma$, our phase $\eta$ has a simple combinatorial structure:

$$\eta(\gamma, \delta; \gamma', \delta') = e^{i\sigma(\delta,\delta')}e^{-i\sigma(\delta,\gamma')}e^{-i\sigma(\gamma,\delta')}e^{i\sigma(\gamma,\gamma')}. \tag{3.3}$$

In the case of localized charges, the intertwiners lie in the observable net as a consequence of duality and the category of intertwiners and automorphisms then acquires the structure of a tensor category. The "tensor product" of automorphisms is just composition of automorphisms and needs no special symbol. Given intertwiners $V \in (\gamma, \delta)$ and $V' \in (\gamma', \delta')$ their "tensor product" is defined by

$$V \times V' := V\gamma(V') \in (\gamma\gamma', \delta\delta'). \tag{3.4}$$

This is a tensor category since if $U \in (\beta, \gamma)$, and $U' \in (\beta', \gamma')$ then

$$U \times U' \circ V \times V' = (U \circ V) \times (U' \circ V'). \tag{3.5}$$

Statistics manifests itself as a permutation symmetry $\varepsilon$ for this category. Thus for each pair $\gamma, \gamma'$ of objects we have an $\varepsilon(\gamma, \gamma') \in (\gamma\gamma', \gamma'\gamma)$ such that

$$\varepsilon(\delta, \delta') \circ V \times V' = V' \times V \circ \varepsilon(\gamma, \gamma'), \quad V \in (\gamma, \delta), V' \in (\gamma', \delta'), \tag{3.6}$$

$$\varepsilon(\gamma', \gamma) \circ \varepsilon(\gamma, \gamma') = 1_{\gamma\gamma'}, \tag{3.7}$$

$$\varepsilon(\gamma\delta, \gamma') = \varepsilon(\gamma, \gamma') \times 1_\delta \circ 1_\gamma \times \varepsilon(\delta, \gamma'). \tag{3.8}$$

Sectors described by automorphisms are referred to as *simple* sectors because the composition law of sectors takes on a simple form making the set of sectors into a discrete group whose dual is then a compact Abelian group which

is the global gauge group when all sectors are simple sectors. In general, in superselection theory we have to deal not just with automorphisms but with endomorphisms and their intertwiners and the tensor category has a more complicated structure not relevant to our present discussion.

Returning to our model, although the free massless field satisfies duality we cannot conclude that the intertwiners lie in the observable net $\mathfrak{A}$ because our automorphisms are localized only relative to $\mathfrak{A}_0$. Nonetheless, after extending our symplectic form to $\mathcal{L}_\Gamma$, there is an obvious tensor category of automorphisms and intertwiners in our case, too. Given $U \in (\gamma, \delta)$ and $U' \in (\gamma', \delta')$, we set

$$U \times U' := U\gamma(U') := e^{i\sigma(\gamma,\delta'-\gamma')}UU'. \tag{3.9}$$

Together with this tensor product structure there is another phase relevant to a discussion of statistics, namely

$$U \times U' \circ {U'}^{-1} \times U^{-1} = e^{i\sigma(\delta,\delta')}e^{-i\sigma(\gamma,\gamma')}. \tag{3.10}$$

The form of this equation shows that our tensor category has a permutation symmetry $\varepsilon$ given by

$$\varepsilon(\gamma, \gamma') = e^{-i\sigma(\gamma,\gamma')}, \tag{3.11}$$

where the phase is understood as a self–intertwiner of $\gamma\gamma' = \gamma'\gamma$.

Summing our results up, we conclude:

**Theorem 2.** *The model has a symmetric tensor category describing a group of simple sectors isomorphic to the discrete group $\mathbb{R}$. The objects are the elements of $\Gamma$, the arrows are the intertwiners between the associated representations of $\mathfrak{A}$. The tensor structure and the permutation symmetry $\varepsilon$ are determined by the symplectic form $\sigma$.*

If we interpret the permutation symmetry as statistics, this means Bose statistics since the phase is one when $\gamma = \gamma'$. To understand why this should indeed be regarded as statistics, we must examine the asymptotic behaviour of the symplectic form.

## 4 Asymptotics

We recall that, in the case of strictly localized charges and $s > 1$, intertwiners $U \in (\gamma, \delta)$ and $U' \in (\gamma', \delta')$ commute if $\gamma$ and $\delta$ are localized in one double cone and $\gamma'$ and $\delta'$ in a spacelike separated double cone. We have $U\times U' = U'\times U$ even under the weaker condition that $\gamma$ and $\gamma'$ are spacelike separated and $\delta$ and $\delta'$ are spacelike separated. This simple situation does not prevail in our model so we investigate the spacelike asymptotic behaviour of these commutation properties and, as we have seen, it suffices to consider the behaviour of the symplectic form $\sigma$. Thus we consider the asymptotic dependence of $\sigma(\gamma_a, \gamma'_b)$ on the translations $a$ and $b$. Of course, this expression depends only on $a - b$, but we prefer to use the more symmetric form. Writing

$$\gamma = i\omega^{-\frac{3}{2}}g + \omega^{-\frac{1}{2}}h, \quad g, h \in \mathcal{D}(\mathbb{R}^s), \tag{4.1}$$

we get a sum of four terms:

$$\sigma(\gamma_a, \gamma_b') = \sigma(i\omega^{-\frac{3}{2}} g_a, i\omega^{-\frac{3}{2}} g_b') + \sigma(i\omega^{-\frac{3}{2}} g_a, \omega^{-\frac{1}{2}} h_b') + \\ +\sigma(\omega^{-\frac{1}{2}} h_a, i\omega^{-\frac{3}{2}} g_b') + \sigma(\omega^{-\frac{1}{2}} h_a, \omega^{-\frac{1}{2}} h_b') \tag{4.2}$$

to be treated separately. The variables involving $h$ and $h'$ are in $\mathcal{L}$ so that the final term vanishes exactly whenever the variables have spacelike separated supports. The second term,

$$\sigma(i\omega^{-\frac{3}{2}} g_a, \omega^{-\frac{1}{2}} h_b') = -\int d^s\vec{p}\, \omega^{-2} \tilde{g}(-\vec{p}) \tilde{h}'(\vec{p}) \cos\omega(a^0 - b^0) e^{i\vec{p}\cdot(\vec{a}-\vec{b})}, \tag{4.3}$$

decays like $|\vec{a} - \vec{b}|^{-1}$. Set

$$Z(x) := \int d^s\vec{p}\, \omega^{-2} \tilde{g}(-\vec{p}) \tilde{h}'(\vec{p}) \cos\omega x^0 e^{i\vec{p}\cdot\vec{x}}, \tag{4.4}$$

then

$$Z(x) = Z(0, \vec{x}) + \int_0^{x^0} dt\, \dot{Z}(t, \vec{x}). \tag{4.5}$$

Since $\dot{Z}$ is a solution of the wave equation with compact Cauchy data, the integral does not contribute when $|\vec{x}| > |x^0|+$ const. and the first term can be written:

$$Z(0, \vec{x}) = \text{const} \int d^s\vec{y}\, (g * h')(\vec{y})\, |\vec{x} - \vec{y}|^{2-s}. \tag{4.6}$$

Hence $Z(x_0, \vec{x}) = \mathrm{O}(|\vec{x}|^{2-s})$ for $|\vec{x}| > |x^0|+$ const. The term involving $g'$ and $h$ is of exactly the same type and thus decays in the same way.

The remaining term is

$$\sigma(i\omega^{-\frac{3}{2}} g_a, i\omega^{-\frac{3}{2}} g_b') = \int d^s\vec{p}\, \omega^{-3} \tilde{g}(-\vec{p}) \tilde{g}'(\vec{p}) \sin\omega(a^0 - b^0) e^{-i\vec{p}\cdot(\vec{a}-\vec{b})}. \tag{4.7}$$

Setting

$$Y(x) := \int d^s\vec{p}\, \omega^{-3} \tilde{g}(-\vec{p}) \tilde{g}'(\vec{p}) \sin\omega x^0 e^{-i\vec{p}\cdot\vec{x}}, \tag{4.8}$$

and writing

$$Y(x) = Y(0, \vec{x}) + x^0 \dot{Y}(0, \vec{x}) + \int_0^{x^0} dt \int_0^t ds\, \ddot{Y}(s, \vec{x}), \tag{4.9}$$

we see that the first summand vanishes. $\ddot{Y}$ is a solution of the wave equation with compact Cauchy data and hence the integral vanishes for sufficiently spacelike $x$. Finally,

$$x^0\, \dot{Y}(0, \vec{x}) = c\tilde{g}(0)\tilde{g}'(0)\, \frac{x^0}{|\vec{x}|^{s-2}} + \mathrm{O}(\frac{x^0}{|\vec{x}|^{s-1}}), \tag{4.10}$$

where $c$ is a non-zero constant. Summing up what we have learned, we have

**Theorem 3.** *The asymptotic behaviour of the symplectic form for $s = 3$ as $a - b$ tends spacelike to infinity is*

$$\sigma(\gamma_a, \gamma'_b) = \frac{1}{4\pi}\, qq' \frac{|a^0 - b^0|}{|\vec{a} - \vec{b}|} + o(1). \tag{4.11}$$

*Here $q = \int d^s\vec{x} g$ and $q' = \int d^s\vec{x} g'$ are the charges carried by $\gamma$ and $\gamma'$.*

In this way we see that intertwiners will commute asymptotically and the cross product will commute asymptotically provided we go spacelike to infinity in such a way that

$$\frac{a^0 - b^0}{|\vec{a} - \vec{b}|} \to 0. \tag{4.12}$$

## 5 Interpretation

After the computations on the asymptotic behaviour of the symplectic form in the last section, we show how to arrive at the tensor category reasoning just in terms of the representations defined by the elements of $\Gamma$ and the corresponding intertwiners without making use of an ad hoc, if natural, extension of the symplectic form from $\mathcal{L}$ to $\mathcal{L}_\Gamma$.

Given $\gamma$, $\delta$, $\delta' \in \Gamma$ and $V \in (\delta, \delta')$, we wish to give a meaning to $\gamma(V)$ as an intertwiner from $(\gamma\delta, \gamma\delta')$. In the case of localized charges it suffices to define

$$\gamma(V) := U^* V U, \tag{5.1}$$

where $U \in (\gamma, \gamma')$ is unitary and $\gamma'$ is localized spacelike to $V$. The physical idea behind this construction is to regard $U$ as an operation of transferring charge. Thus the formula:

$$\gamma(V) := \lim_a U_a^* V U_a, \quad U_a \in (\gamma, \gamma_a), \tag{5.2}$$

where the limit is taken as $a$ tends spacelike to infinity, would be valid for any intertwiner $V$ and we have created the charge carried by $\gamma$ by transferring it from spacelike infinity.

We will use the same formula to define $\gamma(V)$, but now requiring that $\frac{a^0}{|\vec{a}|} \to 0$. In fact, in our model,

$$U_a^* V U_a = \eta(\delta, \delta'; \gamma, \gamma_a) V \tag{5.3}$$

and expressing $\eta$ in terms of the symplectic form $\sigma$ and using its asymptotic behaviour, we conclude that

$$U_a^* V U_a \to e^{i\sigma(\delta - \delta', \gamma)} V. \tag{5.4}$$

But this means that our new definition of $\gamma(V)$ coincides with the old definition and hence serves to turn our category of automorphisms and intertwiners into a tensor category.

The same line of reasoning leads us to a definition of statistics. We know that in the case of strictly localized charges the permutation symmetry $\varepsilon$ describing the statistics can be characterized by requiring that

$$\varepsilon(\gamma,\delta) = (V \times U)^{-1} \circ (U \times V), \tag{5.5}$$

when $U \in (\gamma,\gamma')$, $V \in (\delta,\delta')$ are unitaries and $\gamma'$ and $\delta'$ are spacelike separated. Hence we can try defining the statistics operator by

$$\varepsilon(\gamma,\delta) = \lim_a (V_b \times U_a)^{-1} \circ U_a \times V_b, \tag{5.6}$$

where $U_a \in (\gamma,\gamma_a)$ and $V_b \in (\delta,\delta_b)$ are unitary. If we compute in our model, we find

$$\varepsilon(\gamma,\delta) = \lim_a e^{i\sigma(\delta,\gamma)} e^{-i\sigma(\delta_b,\gamma_a)}. \tag{5.7}$$

Using the asymptotic properties of the symplectic form, we find

$$\varepsilon(\gamma,\delta) = e^{-i\sigma(\gamma,\delta)} \tag{5.8}$$

Note that we might equally well have kept $a$ fixed and sent $b$ to spacelike infinity or have sent $a$ and $b$ to spacelike infinity in opposite directions. So we have verified that the symmetric tensor category and hence, in particular, the statistics, may be derived by physically motivated conditions of an asymptotic nature.

The importance of being able to describe the superselection structure in terms of a symmetric tensor category is that it opens the way to constructing a field net with normal commutation relations acted on by a compact gauge group whose fixed–point net is the observable net. This construction was first carried out for simple sectors in [16] and in the case of general localized charges in [7]. We do not need to discuss how the scheme should be modified to embrace the setting of our model since we get the field net as the net of von Neumann algebras in the vacuum sector generated by $\mathcal{L}_\Gamma$ with the extended symplectic form.

## 6 Outlook

At the first sight, it might appear that our model has many special features rendering the above analysis possible. For example, we were able to verify the existence of norm limits by examining the behaviour of certain phases. However, the intertwining spaces will be one–dimensional whenever we look at sectors generated by automorphisms. Although these automorphisms will not commute, in general, as automorphisms generated by Weyl operators do, they may have sufficient commutativity properties. In the case of localized charges, automorphisms localized spacelike to one another commute and there are generalizations to the case of localization relative to a suitably large subnet. In these cases, our phases will be defined whenever the automorphisms in question have the appropriate spacelike localization properties and the asymptotic norm convergence reduces to the convergence of these phases, just as in the model.

If we do have convergence to appropriate limits then we get a symmetric tensor category of automorphisms and intertwiners describing the corresponding superselection structure. This will be discussed elsewhere.

Thus from a technical point of view we might say that, although our model has simplifying features allowing an easy complete analysis of the sectors in question, the analysis itself applies to a whole class of theories. It is not clear how many physically interesting theories would fall within this class. Yet there are many models having charges localized on some large subnet and it would seem that electric and magnetic charges should exhibit this type of behaviour.

## References


[1] S. Doplicher, R. Haag and J.E. Roberts *Local Observables and Particle Statistics, I*, Commun. Math. Phys. **23** (1971), 199–230, and *II*, Commun. Math. Phys. **35** (1974), 49–85 .

[2] D. Buchholz and K. Fredenhagen *Locality and the Structure of Particle States*, Commun. Math. Phys. **84** (1982), 1–54.

[3] J. Fröhlich, G. Morchio and F. Strocchi *Charged Sectors and Scattering States in Quantum Electrodynamics*, Ann. Phys. **119** (1979), 241-284.

[4] D. Buchholz *Gauss' Law and the Infraparticle Problem*, Phys. Lett. **B174** (1986), 331-334.

[5] F. Strocchi *Selected Topics on the General Properties of Quantum Field Theory*, World Scientific 1993.

[6] D. Buchholz *The Physical State Space of Quantum Electrodynamics*, Commun. Math. Phys. **85** (1982), 49–71.

[7] S. Doplicher and J.E. Roberts *Why there is a Field Algebra with a Compact Gauge Group Describing the Superselection Structure in Particle Physics*, Commun. Math. Phys. **131** (1990), 51–107.

[8] J. Lopuszanski *An Introduction to Symmetry and Supersymmetry in Quantum Field Theory* World Scientific: Singapore 1991

[9] K. Symanzik *Lectures on Lagrangian Quantum Field Theory*, Desy Report T-71/1.

[10] J. Fröhlich *The Charged Sectors of Quantum Electrodynamics in a Framework of Local Observables*, Commun. Math. Phys. **66** (1979), 223-265.

[11] R. Haag *Local Quantum Physics*, Springer: Berlin, Heidelberg, New York 1992.

[12] J. M. Jauch, F. Rohrlich *The Theory of Photons and Electrons* Springer: Heidelberg 1980.

[13] F. Acerbi, G. Morchio and F. Strocchi *Theta Vacua, Charge Confinement and Charge Sectors from Nonregular Representations of CCR Algebras* Lett. Math. Phys. **27** (1993), 1–11; *Infrared Singular Fields and Non–regular Representations of CCR Algebras*, Jour. Math. Phys. **34** (1993), 899–914.

[14] S. Doplicher and J.E. Roberts *A New Duality Theory for Compact Groups*, Inventiones Math. **98** (1989), 157–218.

[15] D. Buchholz, S. Doplicher, R. Longo, J.E. Roberts: *A New Look at Goldstone's Theorem*, Rev. Math. Phys. Special Issue (1992), 49-83.

[16] S. Doplicher, R. Haag, J.E. Roberts: *Fields, Observables and Gauge Transformations II*, Commun. Math. Phys. **15** (1969), 173-200.